General Polynomials

1. Prove that $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}=0$.

Hence, or otherwise, show that the expression $\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$ is a perfect square and find its square root.

2. Prove that if $a_0k^3 + a_1k^2 + a_2k + a_3 = 0$, then $x - k$ is a factor of $a_0x^3 + a_1x^2 + a_2x + a_3$.
Find the four factors of the first degree in x, y, z of $x^3 (y - z) + y^3 (z - x) + z^3 (x - y)$.

3. Find the factors of:

(a) $(b + c)^3 (b - c) + (c + a)^3 (c - a) + (a + b)^3 (a - b)$.

(b) $(b - c)(c + a - b)(a + b - c) + (c - a)(a + b - c)(b + c - a) + (a - b)(b + c - a)(c + a - b)$.

4. The roots of the equation $x^3 + 3x^2 - 2 = 0$ are $-1, a, b$, where $a \neq b$. Prove that there exists a polynomial $f(x) = px^2 + qx + r$ such that $f(1) = 1$, $f(a) = b$ and $f(b) = a$.

5. Show that $ax^3 + bx^2 + cx + d$ is a perfect cube if and only if $b^3 = 27a^2d$ and $c^3 = 27ad^2$.

6. **(a)** Find by the Principle of Undetermined Coefficients the sum $S(n) = 1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots + n(n+1)$.
(Hint: Assume that $S(n) = A + Bn + Cn^2 + Dn^3 + En^4 + \ldots$, find $S(n+1) - S(n)$,
and apply the Principle of Undetermined Coefficients to find $A, B, C, D, E, \ldots$)

(b) Find by the Principle of Undetermined Coefficients the sum $S(n) = 1^3 + 3^3 + 5^3 + 7^3 + \ldots$ to n terms.

7. Find a and b so that $x^3 + ax^2 + 11x + 6$ and $x^3 + bx^2 + 14x + 8$ may have a common factor of the form $x^2 + px + q$.

8. A polynomial in a variable x is said to be a zero polynomial if it is identically zero. Let f and g be polynomials in x. Prove that $f^2 + x g^2$ is a zero polynomial, the f and g are both zero polynomials.

9. The i-th elementary symmetric function $s_i (x_1, x_2, \ldots , x_n)$ in the indeterminates $x_1, x_2, \ldots , x_n$ is defined as the coefficient of y^{n-i} of the polynomial $(y + x_1)(y + x_2) \ldots (y + x_n)$.
Evaluate $s_i (x_1, x_2, x_3, x_4)$ for $i = 1, 2, 3, 4$.

10. **(a)** If $f(x) = \sum_{r=1}^{3n} a_r x^r$, evaluate $f(x) + f(\omega x) + f(\omega^2 x)$, where ω is the complex cube root of 1 .

(b) Calculate similarly $f(x) + \omega f(\omega x) + \omega^2 f(\omega^2 x)$ and $f(x) + \omega^2 f(\omega x) + \omega f(\omega^2 x)$.

11. If a monic polynomial, $a(x)$ with integer coefficients is the product of two monic polynomials $b(x), c(x)$ with rational coefficients, prove that the coefficients in $b(x), c(x)$ must be actually be integer.
(Note : A monic polynomial is one with one as coefficient in the term with highest power.)

12. **(a)** Show that if $a(x)f(x) + g(x) = 0$, $a(x) \neq 0$ and $g(x)$ is zero or has lower degree than $a(x)$,

then $f(x) = 0$ and $g(x) = 0$.

(b) If $b(x) = a(x)q(x) + r(x)$ and $b(x) = a(x)Q(x) + R(x)$, where $r(x)$ and $R(x)$ both have degree less than that of $a(x)$, then $q(x) = Q(x)$ and $r(x) = R(x)$.

13. **(a)** Prove that there exist polynomials $M(x)$, $N(x)$ such that $M(x)f(x) + N(x)g(x)$ is the H.C.F. of $f(x)$, $g(x)$.

(b) Prove that $f(x), g(x)$ are relatively prime and $R(x), S(x)$ are polynomials such that $R(x)f(x) = S(x)g(x)$, then $f(x)$ divides $S(x)$ and $g(x)$ divides $R(x)$.

14. **(a)** Prove that if $a^2(b - c) + b^2 (c - a) + c^2 (a - b) = 0$ then $(a^n - b^n)(b^n - c^n)(c^n - a^n) = 0$.

(b) Prove that if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, then $\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{(a+b+c)^{2n+1}}$.

where n is a positive integer.

(c) Prove that if $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1$, then

$\left(\frac{b^2+c^2-a^2}{2bc}\right)^{2n+1}+\left(\frac{c^2+a^2-b^2}{2ca}\right)^{2n+1}+\left(\frac{a^2+b^2-c^2}{2ab}\right)^{2n+1}=1$, where n is a positive integer.

15. Resolve in factors:

(a) $ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)$

(b) $(x - y)^5 + (y - z)^5 + (z - x)^5$

(c) $(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4$.

16. If $f(x)$, a polynomial in x, is divided by $(x - a)(x - b)$, prove that the remainder is

$\frac{f(a)-f(b)}{a-b}x+\frac{af(b)-bf(a)}{a-b}$.

17. Prove that $(x^2 + x + 1)^n - (x^2 - x - 1)^n$ is divisible by $1 + x$.

Hence or otherwise show that $111^n - 89^n$ is divisible by 11.

18. **(a)** Express $x^6 + 3x^4 - 7$ as a polynomial in $x^2 + 1$.

(b) Let $P(x) = x^8 + x^7 + 6x^6 + 3x^5 + 12x^4 + 4x^2 - 7x - 13$

$Q(x) = (x + 2) (x^2 + 1)^4$

Find a polynomial $g(x)$ which satisfies the identity : $P(x) \equiv (x^2 + 1)^4 + g(x) (x + 2)$.

Hence resolve $\frac{P(x)}{Q(x)}$ into partial fractions.